1)

$$X_{rad}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j f t}\, dt$$

$$X_{rad}(2\pi f) = \int_{-\infty}^{\infty} x(t)\, e^{-j(2\pi f)t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j f t}\, dt = X(f)$$

$$X\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j \left(\frac{\omega}{2\pi}\right)t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X_{rad}(\omega)$$

3.a) (7.1-3)

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right) \qquad X_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

$$\mathcal{F}\{x_e(t)\} = \frac{1}{2}\mathcal{F}\{x(t)\} + \frac{1}{2}\mathcal{F}\{x(-t)\} = \frac{1}{2}X(f) + \frac{1}{2}X(-f) =$$

$$= \text{even}\left(X(f)\right)$$

$$\mathcal{F}\{x_o(t)\} = \mathcal{F}\{\tfrac{1}{2}(x(t) - x(-t))\} = \frac{1}{2}\left(X(f) - X(-f)\right) =$$

$$= \text{odd}\left(X(f)\right)$$

What the book doesn't say, and what I forgot to point out, is that $x(t)$ is assumed to be real. This means that $X(f)$ is hermitian. Hermitian functions have even real parts and odd imaginary parts (alternatively, the even part is pure real and the odd part is pure imaginary).

$X(f)$ is hermitian $\leftrightarrow$ $X(f) = X^*(-f)$

If $X_o(f)$ is odd and hermitian: $\underbrace{X_o(f) = X_o^*(-f)}_{\text{hermitian}}$ $\underbrace{-X_o(f) = X_o(-f)}_{\text{odd}}$

replace f with $-f$: $X_o(-f) = \underbrace{X_o^*(f) = -X_o(f)}_{\text{pure imaginary}}$

$\therefore$ the odd part of hermitian $X(f)$ is pure imaginary.

If $X(f)$ is even and hermitian: $X_e(f) = X_e^*(-f)$ and $X_e(f) = X_e(-f)$

$$\rightarrow \quad X_e(-f) = \underbrace{X_e^*(f) = X_e(f)}_{\text{pure real}}$$

$\therefore$ the odd part of hermitian $X(f)$ is pure real.

So, $\text{even}\left(X(f)\right) = \text{Re}\left(X(f)\right)$ and $\text{odd}\left(X(f)\right) = j\,\text{Im}\left(X(f)\right)$

2.b)

$$\text{even}(u(t)) = \tfrac{1}{2}(u(t) + u(-t)) = \tfrac{1}{2}$$

$$\text{odd}(u(t)) = \tfrac{1}{2}(u(t) - u(-t)) = \tfrac{1}{2}\,\text{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \tfrac{1}{2}\delta(f) + \frac{1}{2\pi j f} = U(f) \rightarrow \begin{array}{l} \text{Re}(U(f)) = \tfrac{1}{2}\delta(f) \\ \text{Im}(U(f)) = -\frac{1}{2\pi f} \end{array}$$

$$\mathcal{F}\{\tfrac{1}{2}\} = \tfrac{1}{2}\delta(f) = \text{Re}(U(f))$$

$$\mathcal{F}\{\tfrac{1}{2}\,\text{sgn}(t)\} = \frac{1}{2}\frac{1}{\pi j f} = -j\left(\frac{1}{2\pi f}\right) = j\,\text{Im}(U(f))$$

Note: the Fourier transform only exists if $a \geq 0$.

$$x(t) = e^{-at}u(t)$$

$$\text{even}(x(t)) = \tfrac{1}{2}(e^{-at}u(t) + e^{+at}u(-t)) = \tfrac{1}{2}e^{-|at|}$$

$$\text{odd}(x(t)) = \tfrac{1}{2}(e^{-at}u(t) - e^{+at}u(-t)) = \tfrac{1}{2}\,\text{sgn}(t)\,e^{-|at|}$$

$$\mathcal{F}\{e^{-at}u(t)\} = a\frac{1}{1+2\pi j f a} = a\frac{1}{1+2\pi j a f}\,\frac{1-2\pi j a f}{1-2\pi j a f} = \frac{a(1-2\pi j a f)}{|1+2\pi j a f|^2}$$

$$= \frac{a}{1+(2\pi a f)^2} + -j\,\frac{2\pi a^2 f}{1+(2\pi a f)^2}$$

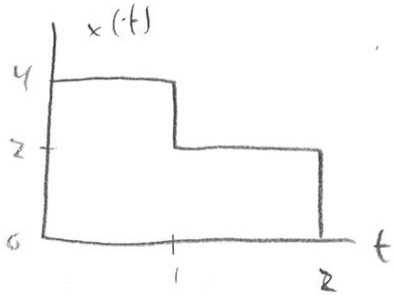
$$\mathcal{F}\{\tfrac{1}{2}e^{-|at|}\} = \frac{1}{2}a\frac{2}{1+(2\pi a f)^2} = \frac{a}{1+(2\pi a f)^2} = \text{Re}(X(f))$$

$$\mathcal{F}\{\tfrac{1}{2}(e^{-at}u(t) - e^{+at}u(-t))\} = \frac{1}{2}\frac{a}{1+2\pi j a f} - \frac{1}{2}\frac{a}{1-2\pi j f}$$

$$= \frac{1}{2}\frac{a}{1+2\pi j f a} - \frac{1}{2}\left(\frac{a}{1+2\pi j f}\right)^*$$

$$= j\,\text{Im}(X(f))$$

3.a 7.1-5



$$x(t) = 4\Pi(t-\tfrac{1}{2}) + 2\Pi(t-\tfrac{3}{2})$$
$$= 2\Pi(t-\tfrac{1}{2}) + 2\Pi\left(\tfrac{t-1}{2}\right)$$

$$X(f) = 4e^{-2\pi jf(\frac{1}{2})}\operatorname{sinc}(f) + 2e^{-2\pi jf(\frac{3}{2})}\operatorname{sinc}(f)$$
$$= \left(4e^{-2\pi jf(\frac{1}{2})} + 2e^{-2\pi jf(\frac{3}{2})}\right)\operatorname{sinc}(f)$$

$$\Pi\left(\tfrac{t}{2}\right) \xrightarrow{\mathcal{F}} 2\operatorname{sinc}(2f)$$
$$\Pi\left(\tfrac{t-1}{2}\right) \xrightarrow{\mathcal{F}} 2e^{-2\pi jf}\operatorname{sinc}(2f)$$
$$X(f) = 2e^{-2\pi jf(\frac{1}{2})}\operatorname{sinc}(f) + 4e^{-2\pi jf}\operatorname{sinc}(2f)$$

Let's make sure they're equal.

$$e^{-2\pi jf}\operatorname{sinc}(2f) = e^{-2\pi jf}\frac{\sin(2\pi f)}{2\pi f} = e^{-2\pi jf}\frac{1}{2j}\frac{1}{2\pi f}\left(e^{2\pi jf} - e^{-2\pi jf}\right)$$

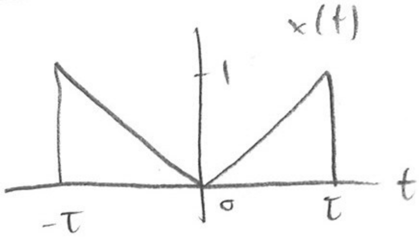
$$e^{+2\pi jf} - e^{-2\pi jf} = \left(e^{\pi jf}\right)^2 - \left(e^{-\pi jf}\right)^2 = \left(e^{\pi jf} + e^{-\pi jf}\right)\left(e^{\pi jf} - e^{-\pi jf}\right)$$

$$\frac{e^{-2\pi jf}\left(e^{2\pi jf} - e^{-2\pi jf}\right)}{2j\,2\pi f} = \frac{1}{2}e^{-2\pi jf}\left(e^{\pi jf} + e^{-\pi jf}\right)\frac{e^{\pi jf} - e^{-\pi jf}}{(2j)\pi f}$$
$$= \frac{1}{2}e^{-2\pi jf}\left(e^{\pi jf} + e^{-\pi jf}\right)\operatorname{sinc}(f) = \frac{1}{2}e^{-\pi jf}\operatorname{sinc}(f) + \frac{1}{2}e^{-3\pi jf}\operatorname{sinc}(f)$$

$$\rightarrow X(f) = \left(2e^{-2\pi jf(\frac{1}{2})} + 2e^{-\pi jf} + 2e^{-3\pi jf}\right)\operatorname{sinc}(f)$$
$$= \left(4e^{-2\pi jf(\frac{1}{2})} + 2e^{-2\pi jf(\frac{3}{2})}\right)\operatorname{sinc}(f)$$

Good, they are equal.

3, b)



$$x(t) = \Pi\left(\frac{t}{2\tau}\right) - \Lambda\left(\frac{t}{\tau}\right)$$

$$\mathcal{F}\{x(t)\} = 2\tau \,\mathrm{sinc}(2\tau f) - \tau \,\mathrm{sinc}^2(\tau f)$$

4)

First, note that $e^{j\pi t^2} = \cos(\pi t^2) + j\sin(\pi t^2)$ by Euler's identity.

Then recall that $\int_{-\infty}^{\infty} x(t)\,dt = X(0)$.

$$\int_{-\infty}^{\infty} e^{j\pi t^2}\,dt = \frac{1+j}{\sqrt{2}} e^{-j\pi 0^2} = \frac{1+j}{\sqrt{2}} = \int_{-\infty}^{\infty} \cos(\pi t^2)\,dt + j\int_{-\infty}^{\infty} \sin(\pi t^2)\,dt$$

Now, we could just make a change of variables and get the integrals we want, but instead, we will use the dilation rule, to find a different Fourier transform.

$$e^{j\pi t^2} \xrightarrow{\tilde{\mathcal{F}}} e^{-j\pi f^2}\frac{1+j}{\sqrt{2}}$$

$$e^{jt^2} = \underbrace{e^{j\pi\left(\frac{t}{\sqrt{\pi}}\right)^2}}_{x\left(\frac{t}{\sqrt{\pi}}\right)} \xrightarrow{\tilde{\mathcal{F}}} \underbrace{\sqrt{\pi}\,\frac{1+j}{\sqrt{2}}\,e^{-j\pi(\sqrt{\pi}f)^2}}_{\sqrt{\pi}\,X(f\sqrt{\pi})}$$

$$\int_{-\infty}^{\infty} e^{jt^2}\,dt = \int_{-\infty}^{\infty} \cos(t^2)\,dt + j\int_{-\infty}^{\infty} \sin(t^2)\,dt = \sqrt{\pi}\,\frac{1+j}{\sqrt{2}}\,e^{-j\pi(\sqrt{\pi}f)^2} = \sqrt{\frac{\pi}{2}}(1+j)$$

Equate the real and imaginary parts!

$$\int_{-\infty}^{\infty} \cos(t^2)\,dt = \sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \sin(t^2)\,dt = \sqrt{\frac{\pi}{2}}$$

Note: we can't use Parseval's rule $\left(\int_{-\infty}^{\infty} |x|^2\,dt = \int_{-\infty}^{\infty} |X|^2\,df\right)$. This is because $e^{+j\pi t^2}$ is not square integrable.

$$\int_{-\infty}^{\infty} \left|e^{j\pi t^2}\right|^2 dt = \int_{-\infty}^{\infty} 1\,dt = \infty$$

5.a)

$$\frac{dy}{dt} + y = x \xrightarrow{\mathcal{F}} 2\pi j f\, Y(f) + Y(f) = X(f) \rightarrow Y(f) = X(f) \underbrace{\frac{1}{1+2\pi j f}}_{H(f)}$$

Using the convolution rule, $\mathcal{F}^{-1}\{Y(f)\} = \mathcal{F}^{-1}\{X(f)H(f)\}$

$$= (x * h)(t)$$

Fortunately, we found $h(t) = \mathcal{F}^{-1}\{\frac{1}{1+j2\pi f}\}$ in class: $h(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{-\lambda} u(\lambda)\, x(t-\lambda)\, d\lambda$$

5.b) $x(t) = \Pi(t - \frac{1}{2})$

$$\Pi(t) \xrightarrow{\mathcal{F}} \operatorname{sinc}(f)$$

$$\Pi(t-\tfrac{1}{2}) \xrightarrow{\mathcal{F}} e^{-2\pi j f (\frac{1}{2})} \operatorname{sinc}(f) =$$

$$Y(f) = X(f)H(f) = \frac{1}{1+2\pi j f} e^{-2\pi j f(\frac{1}{2})} \operatorname{sinc}(f) = \frac{1}{1+2\pi j f} e^{-\pi j f} \frac{e^{+\pi j f} - e^{-\pi j f}}{2j\pi f}$$

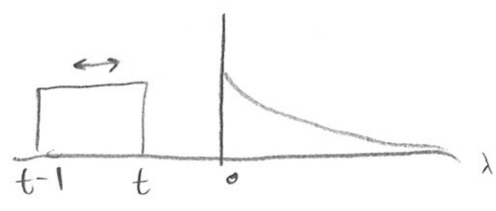
$$= \frac{1}{1+2\pi j f} \frac{1}{2\pi j f}\left(1 - e^{-2\pi j f}\right)$$

$$\frac{1}{1+2\pi j f} \xrightarrow{\mathcal{F}^{-1}} e^{-t}u(t)$$

$$\frac{1}{1+2\pi j f}\frac{1}{2\pi j f} \xrightarrow{\mathcal{F}^{-1}} \int_{-\infty}^{t} e^{-\lambda} u(\lambda)\, d\lambda = \begin{cases} -e^{-\lambda}\Big]_{\lambda=0}^{t} & t > 0 \\ 0 & t < 0 \end{cases} = (1-e^{-t})u(t)$$

$$(1 - e^{-2\pi j f}) \frac{1}{2\pi j f}\frac{1}{1+2\pi j f} \xrightarrow{\mathcal{F}^{-1}} (1-e^{-t})u(t) - (1-e^{-(t-1)})u(t-1) = y(t)$$

Convolution is easier in this case:



$$y(t) = \int_{-\infty}^{\infty} \Pi(t-\lambda-\tfrac{1}{2})\, e^{-\lambda} u(\lambda)\, d\lambda$$

$$= \int_{t-1}^{t} e^{-\lambda} u(\lambda)\, d\lambda = (1-e^{-\lambda})u(\lambda)\Big]_{\lambda=t-1}^{t}$$

$$= (1-e^{-t})u(t) - (1-e^{-(t-1)})u(t-1)$$

6]

We know that $\tilde{\mathcal{F}}\{e^{-\pi t^2}\} = e^{-\pi f^2}$.

$$e^{-t^2} = e^{-\pi\left(\frac{t}{\sqrt{\pi}}\right)^2} \xrightarrow{\tilde{\mathcal{F}}} \sqrt{\pi}\, e^{-\pi(\sqrt{\pi} f)^2} = \sqrt{\pi}\, e^{-\pi^2 f^2}$$

$$\frac{1}{\sqrt{2\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} = \frac{1}{\sqrt{2\sigma^2}} e^{-\left(\frac{t}{\sqrt{2}\sigma}\right)^2} \xrightarrow{\tilde{\mathcal{F}}} \frac{1}{\sqrt{2\sigma^2}} \sqrt{2}\,\sigma \sqrt{\pi}\, e^{-\pi^2(\sqrt{2}\sigma f)^2} = \sqrt{\pi}\, e^{-2\pi^2\sigma^2 f^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \xrightarrow{\tilde{\mathcal{F}}} \frac{1}{\sqrt{\pi}} \sqrt{\pi}\, e^{-2(\pi\sigma f)^2} e^{-2\pi j f\mu} = e^{-2\pi j f\mu} e^{-2(\pi\sigma f)^2}$$

Invoke the convolution rule:

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(t-\mu_1)^2}{2\sigma_1^2}} * \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{\frac{(t-\mu_2)^2}{2\sigma_2^2}} \xrightarrow{\tilde{\mathcal{F}}} e^{-2\pi j f\mu_1} e^{-2(\sigma_1\pi f)^2} \cdot e^{-2\pi j f\mu_2} e^{-2(\sigma_2\pi f)^2}$$

$$e^{-2\pi j f(\mu_1+\mu_2)} e^{-2\pi^2(\sigma_1^2+\sigma_2^2)f^2}$$

$$= e^{-2\pi j f\underbrace{(\mu_1+\mu_2)}_{\mu}} e^{-2\left(\pi\underbrace{\sqrt{(\sigma_1^2+\sigma_2^2)}}_{\sigma} f\right)^2}$$

$$= e^{-2\pi\mu} e^{-2(\pi\sigma f)^2} \xrightarrow{\tilde{\mathcal{F}}^{-1}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

$$\boxed{\begin{aligned}\mu &= \mu_1+\mu_2\\ \sigma^2 &= \sigma_1^2+\sigma_2^2\end{aligned}}$$

Note: this result is very important in probability theory.